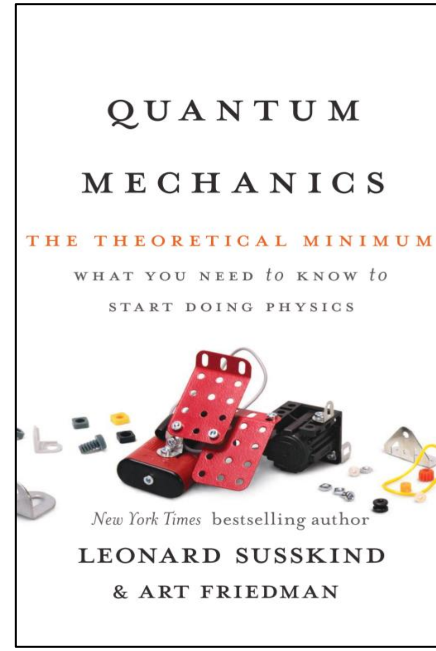
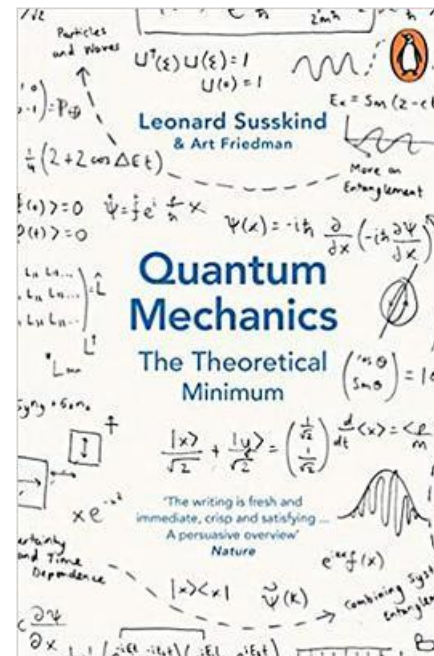
预习通知

- QMTM（前四章）
  - Systems and Experiments
  - Quantum States
  - Principles of Quantum Mechanics
  - Time and Change





预习通知

- 后科普视频

视频链接：https://space.bilibili.com/274326003

Quantum Computing

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Review: Lecture 3

- Classical Deterministic Systems
  - States, dynamics (transition graphs, adjacency matrices)
  - Evolvement
- Probabilistic Systems
  - Probabilistic states and doubly stochastic matrices
- Quantum Systems
  - Quantum states and unitary matrices
  - Comparison of three systems
  - Superposition and measurement

Lecture 4: Basic Quantum Theory

1

Quantum States

- Quantum superposition states
- Case 1: position on a line
- Case 2: single-particle spin system (source: QMTM)
- Complex/probability/transition amplitudes
- keynotes

2

Observables & Measuring

- Observable & Measuring
- Classic physics vs. quantum physics
- The principles (source: QMTM)
- Expected value of observing
- Multiple step observing
- keynotes

3

Dynamics

- The principle (cont.)
- Features of quantum dynamics
- Preview of quantum computation
- Schrödinger equation
- keynotes

（感谢弘毅学堂 2018级宋文卓同学指正此页课程索引号错误）

Supplementary Material

- Greek alphabet

α	β	γ	δ	ε	ζ
Alpha	Beta	Gamma	Delta	Epsilon	Zeta
ν	ξ	ο	π	ρ	σ
Nu	Xi	Omicron	Pi	Rho	Sigma
η	θ	ι	κ	λ	μ
Eta	Theta	Iota	Kappa	Lambda	Mu
τ	υ	φ	χ	ψ	ω
Tau	Upsilon	Phi	Chi	Psi	Omega

感谢弘毅学堂 2020 级陈胤良同学指出Lambda拼写错误

1. Quantum States

- Single photon double slit experiment

视频出处：https://www.bilibili.com/video/BV1ZV411Y7wX?spm_id_from=333.880.my_history.page.click

（感谢弘毅学堂 2018级李梓源同学指正此页单词slit拼写错误）

1. Quantum States

- Single photon double slit experiment

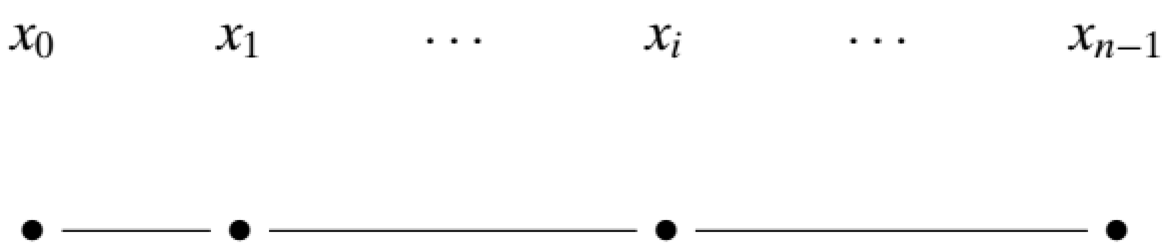
视频出处：https://www.bilibili.com/video/BV1YW411E7vp/?spm_id_from=333.788.recommend_more_video.0

1. Quantum States

- Single photon double slit experiment
  - Phenomenon: interference fringe（干涉条纹）!!
  - Explanation
    - Interference with itself
    - But how? Superposition!->Quantum state

1. Quantum States

- Case 1: positions on line



- Position x_i and state $|x_i\rangle$ (Dirac ket notation)
- State associated column vectors
  - E.g. $|x_0\rangle \longmapsto [1, 0, \ldots .0]^T$

    $|x_1\rangle \longmapsto [0, 1, \ldots .0]^T$

1. Quantum States

- Case 1: positions on line
  - Superposition
    - Linear combination $|\psi\rangle = \sum_{i=0}^{n-1} c_i |x_i\rangle$
      - $c_i \in \mathbb{C}$ is complex amplitudes
    - Vector representation $|\psi\rangle = [c_0,\ c_1,\ \cdots,\ c_{n-1}]^T$
    - Observing probability $p(x_i) = \frac{|c_i|^2}{||\psi\rangle|^2} = \frac{|c_i|^2}{\sum_j |c_j|^2}$
    - Complex amplitudes are probability amplitude if $|\psi\rangle$ is normalized

1. Quantum States

- Case 1: positions on line

Example 4.1.1 Let us assume that the particle can only be at the four points $\{x_0, x_1, x_2, x_3\}$. Thus, we are concerned with the state space $\mathbb{C}^4$. Let us also assume that now the state vector is

$$|\psi\rangle = \begin{bmatrix} -3-i \\ -2i \\ i \\ 2 \end{bmatrix}. \tag{4.7}$$

We shall calculate the probability that our particle can be found at position x_2. The norm of $|\psi\rangle$ is given by

$$||\psi\rangle| = \sqrt{|-3-i|^2 + |-2i|^2 + |i|^2 + |2|^2} = 4.3589. \tag{4.8}$$

The probability is therefore

$$\frac{|i|^2}{(4.3589)^2} = 0.052624. \tag{4.9}$$

□

1. Quantum States

- Case 1: positions on line
  - Kets can be added

$$|\psi\rangle + |\psi'\rangle = (c_0 + c_0')|x_0\rangle + (c_1 + c_1')|x_1\rangle + \cdots + (c_{n-1} + c_{n-1}')|x_{n-1}\rangle$$
$$= [c_0 + c_0', c_1 + c_1', \ldots, c_{n-1} + c_{n-1}']^T. \quad (4.13)$$

  - A ket has complex scalar multiplication

$$c|\psi\rangle = cc_0|x_0\rangle + cc_1|x_1\rangle + \cdots + cc_{n-1}|x_{n-1}\rangle = [cc_0, cc_1, \ldots, cc_{n-1}]^T. \quad (4.14)$$

  - Ket and its complex scalar multiplies describe the same physical state （回忆一下：特征值与特征向量）
    - A ket's length does not matter as far as physics goes

1. Quantum States

■ Case 2: Single-particle spin system

- Stern-Gerlach experiment

视频出处：https://www.bilibili.com/video/BV1ta4y1a7fp?from=search&seid=2882474434643948118&spm_id_from=333.337.0.0

1. Quantum States

- Case 2: Single-particle spin system
  - Stern-Gerlach experiment

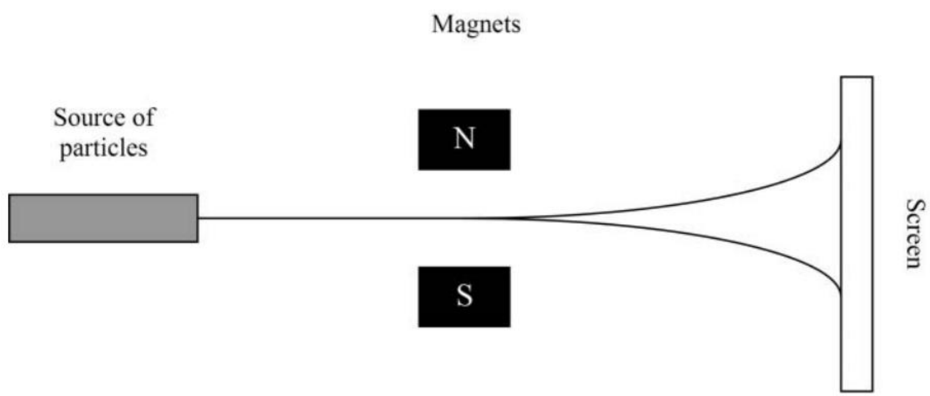
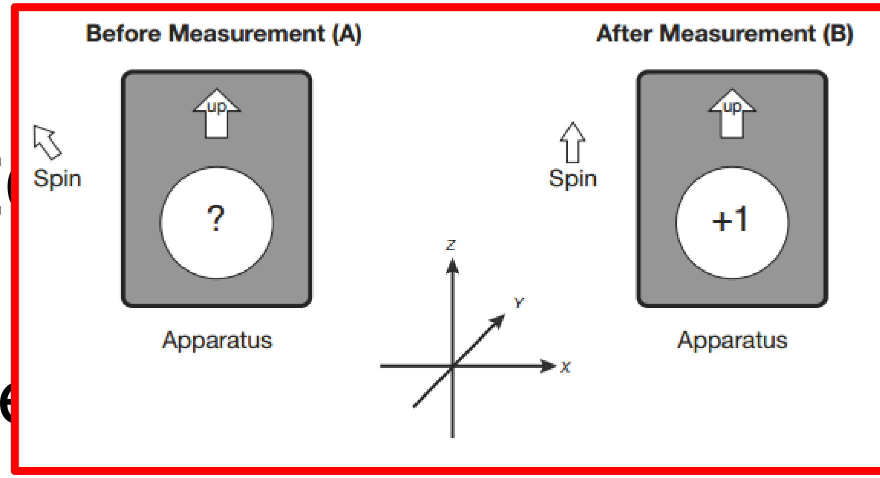


Figure 4.3. The Stern–Gerlach experiment.

- Spin is a intrinsic property （内禀性）
- Discrete spin states (no intermediate)

1. Quantum State



- Case 2: Single-particle
  - Stern-Gerlach experiment （Step 1： 0^{o}）
    - 子图A表示测量之前粒子自旋方向与SGA仪器摆放方位
    - 子图B表示测量之后粒子自旋方向与SGA仪器摆放方位，此时粒子观测为$\sigma_z = +1$，也称为粒子被制备为 (is prepared in the state) 状态$\sigma_z = +1$
    - 如果粒子状态不受扰动，SGA仪器保持测量方位不变，则后续测量会得到相同结果（这解释了前面的“制备”）

1. Quantum States

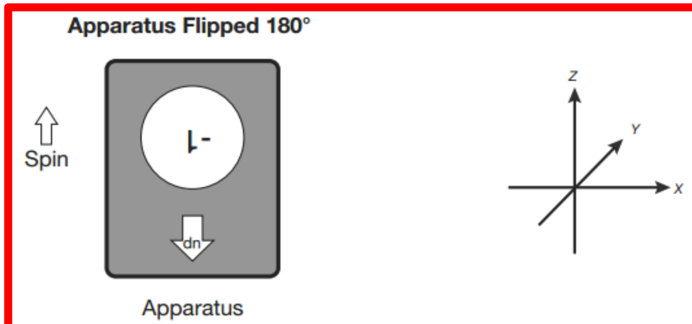


Figure 1.2: The apparatus is flipped without disturbing the previously measured spin. A new measurement results in $\sigma_z = -1$.

- Case 2: Single-particle sp
  - Stern-Gerlach experiment （Step 2：180^o）
    - 如果将SGA仪器倒放，即旋转180^o，且不改变已测得的粒子自旋状态($|u\rangle$)，此时新的测量结果为$\sigma_z = -1$

1. Quantum States

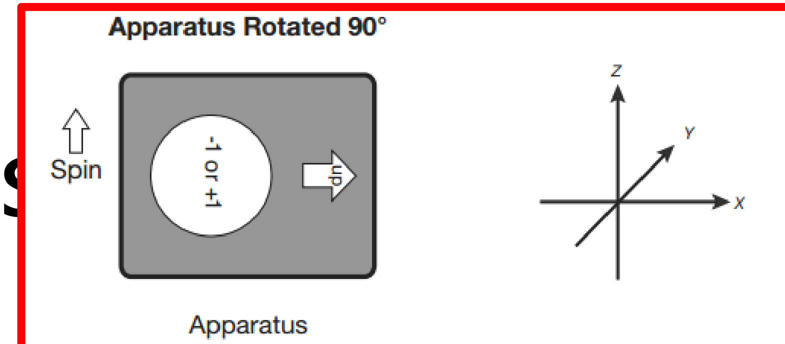


Figure 1.3: The apparatus rotated by 90°. A new measurement results in $\sigma_z = -1$ with 50 percent probability.

- Case 2: Single-particle s
  - Stern-Gerlach experiment （Step 3：90^o）
    - 如果将SGA仪器旋转90^o且不改变已测得的粒子自旋状态($|u\rangle$)，此时新的测量结果为50%的概率得到$\sigma_z = -1$，50%的概率得到$\sigma_z = +1$，期望值为0

1. Quantum States

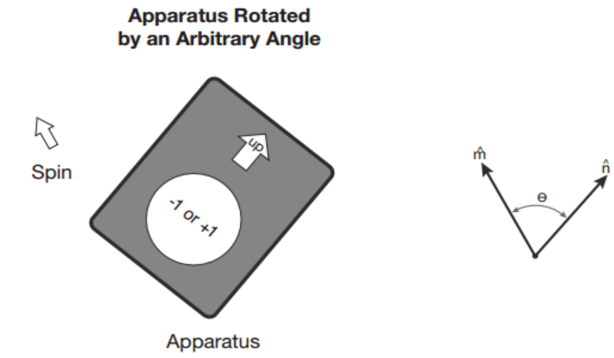


Figure 1.4: The apparatus rotated by an arbitrary angle within the x–z plane. Average measurement result is $\hat{n} \cdot \hat{m}$.

- Case 2: Single-particle spin
  - Stern-Gerlach experiment （Step 4： θ）
    - 如果将SGA仪器旋转 θ 且不改变已测得的粒子自旋状态($|u\rangle$)，此时新的测量结果期望值为$\hat{n} \cdot \hat{m}$ $(= \cos\theta)$
    - 例子：假设 $\theta = \pi/4$

$$\begin{cases} p(+1) \cdot 1 + p(-1) \cdot (-1) = \cos \pi/4 \\ p(+1) + p(-1) = 1 \end{cases}$$

$$\rightarrow \; p(+1) = 85.4\%, \;\; p(-1) = 14.6\%$$

（感谢弘毅学堂2020级郑颖灏同学指正此页θ角度符号书写错误）

1. Quantum States

- Case 2: Single-particle spin system
  - Superposition in the vertical axis

$$|\psi\rangle = c_0|\uparrow\rangle + c_1|\downarrow\rangle$$

Example 4.1.4 Consider a particle whose spin is described by the ket

$$|\psi\rangle = (3-4i)|\uparrow\rangle + (7+2i)|\downarrow\rangle. \tag{4.25}$$

The length of the ket is

$$\sqrt{|3-4i|^2 + |7+2i|^2} = 8.8318. \tag{4.26}$$

Therefore, the probability of detecting the spin of the particle in the up direction is

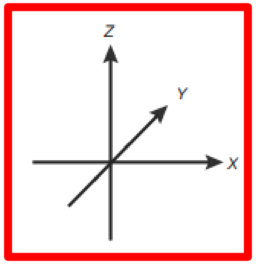
$$p(\uparrow) = \frac{|3-4i|^2}{8.8318^2} = \frac{25}{78}. \tag{4.27}$$

The probability of detecting the spin of the particle in state down is

$$p(\downarrow) = \frac{|7+2i|^2}{8.8318^2} = \frac{53}{78}. \tag{4.28}$$

□

1. Quantum States



- Case 2: Single-particle spin system
  - Representing spin states

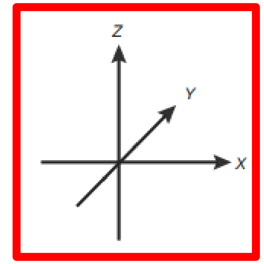
$$|\psi\rangle = \alpha_u |u\rangle + \alpha_d |d\rangle$$

➢ $\alpha_u = \langle u|\psi\rangle$ and $\alpha_d = \langle d|\psi\rangle$ are probability amplitudes

➢ $\begin{cases} p(u) = \alpha_u^\dagger \alpha_u = \langle \psi|u\rangle \langle u|\psi\rangle \\ p(d) = \alpha_d^\dagger \alpha_d = \langle \psi|d\rangle \langle d|\psi\rangle \\ \alpha_u^\dagger \alpha_u + \alpha_d^\dagger \alpha_d = 1 \end{cases}$ are observing probability

（感谢人工智能专业2020级何姜杉同学指正此页关于$\langle u|\psi\rangle$和$\langle \psi|u\rangle$乘积的顺序错误）

1. Quantum States



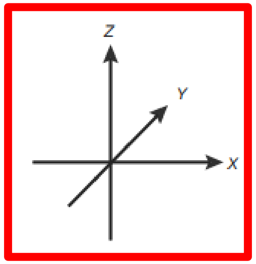
- Case 2: Single-particle spin system
  - Representing spin states (along the x-axis)

recall from Lecture 1, if $\mathcal{A}$ initially prepares $|r\rangle$, and is then rotated to measure σ_z, there will be equal probabilities for *up* and *down*. Thus, $\alpha_u^*\alpha_u$ and $\alpha_d^*\alpha_d$ must both be equal to $\frac{1}{2}$. A simple vector that satisfies this rule is

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle. \quad (2.5)$$

$$\begin{cases} \langle r|l\rangle = 0 \\ \langle l|r\rangle = 0 \end{cases} \Rightarrow |l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle.$$

1. Quantum States



- Case 2: Single-particle spin system
  - Representing spin states (along the y-axis)

$$\langle i|o\rangle = 0.$$

$$\begin{aligned}\langle o|u\rangle\langle u|o\rangle &= \frac{1}{2}\\ \langle o|d\rangle\langle d|o\rangle &= \frac{1}{2}\\ \langle i|u\rangle\langle u|i\rangle &= \frac{1}{2}\\ \langle i|d\rangle\langle d|i\rangle &= \frac{1}{2}.\end{aligned}$$

Equal probabilities for *up* and *down*

$$\begin{aligned}\langle o|r\rangle\langle r|o\rangle &= \frac{1}{2}\\ \langle o|l\rangle\langle l|o\rangle &= \frac{1}{2}\\ \langle i|r\rangle\langle r|i\rangle &= \frac{1}{2}\\ \langle i|l\rangle\langle l|i\rangle &= \frac{1}{2}\end{aligned}$$

Equal probabilities for *left* and *right*

$$\Rightarrow \quad \begin{aligned}|i\rangle &= \frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle\\ |o\rangle &= \frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle\end{aligned}$$

1. Quantum States

- Case 2: Single-particle spin system
  - Inner product
    - Geometric view: overlap
    - Physical view: transition amplitude

In Chapter 2, the inner product was introduced as an abstract mathematical idea. This product turned a vector space into a space with a *geometry*: angles, orthogonality, and distance were added to the canvas. Let us now investigate its physical meaning. The inner product of the state space gives us a tool to compute complex numbers known as **transition amplitudes**, which in turn will enable us to determine how likely the state of the system *before* a specific measurement (start state), will change to another (end state), *after* measurement has been carried out. Let

1. Quantum States

■ Case 2: Single-particle spin system

- Recipe of computing transition amplitude

$$|\psi\rangle = [c_0, c_1, \cdots, c_{n-1}]^T \;\rightarrow\; |\psi'\rangle = [c'_0, c'_1, \cdots, c'_{n-1}]^T$$

➢ Start (normalized) state is $|\psi\rangle = [c_0, c_1, \cdots, c_{n-1}]^T$

➢ End state is the bra vector $\langle\psi'| = |\psi'\rangle^{\dagger} = [\overline{c'_0}, \overline{c'_1}, \cdots, \overline{c'_{n-1}}]$

➢ Transition amplitude is their matrix multiplication

$$\langle\psi'|\psi\rangle = [\overline{c'_0}, \overline{c'_1}, \cdots, \overline{c'_{n-1}}]\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = \sum_{i=0}^{n-1} \overline{c'_i} \times c_i$$

（感谢弘毅学堂 2018级魏瑄同学指正此页关于bra与ket的解释错误，应该是bra-c-ket表示括号）
（感谢弘毅学堂 2018级包云开同学指正此页end state向量第二个元素（脚标为1）遗漏的书写错误）
（感谢弘毅学堂 2020级王康宁同学指正此页bra向量第二个元素脚标应为1的书写错误）

1. Quantum States

- Case 2: Single-particle spin system
  - Transition amplitude=0 <-> orthogonal
  - Orthogonal states are mutually exclusive
    - There are linear independent <-> basis

Note: The transition amplitude between two states may be zero. In fact, that happens precisely when the two states are orthogonal to one another. This simple fact hints at the physical content of orthogonality: orthogonal states are as far apart as they can possibly be. We can think of them as *mutually exclusive alternatives*: for instance, an electron can be in an arbitrary superposition of spin up and down, but after we measure it in the z direction, it will always be *either* up *or* down, never both up *and* down. If our electron was already in the up state before the z direction measurement, it will never transition to the down state as a result of the measurement.

1. Quantum States

- Case 2: Single-particle spin system
  - Transition amplitude as complex/probability amplitude

We can express $|\psi\rangle$ in the basis $\{|b_0\rangle, |b_1\rangle, \ldots, |b_{n-1}\rangle\}$ as

$$|\psi\rangle = b_0|b_0\rangle + b_1|b_1\rangle + \cdots + b_{n-1}|b_{n-1}\rangle. \tag{4.33}$$

We invite you to check that $b_i = \langle b_i|\psi\rangle$ and that $|b_0|^2 + |b_1|^2 + \cdots + |b_{n-1}|^2 = 1$.

It is thus natural to read Equation (4.33) in the following way: each $|b_i|^2$ is the *probability* of ending up in state $|b_i\rangle$ after a measurement has been made.

1. Quantum States

- Case 2: Single-particle spin system

Example 4.1.7 Let us calculate the amplitude of the transition from $|\psi\rangle = [1, -i]^T$ to $|\phi\rangle = [i, 1]^T$. Both vectors have norm $\sqrt{2}$.

We can take their inner product first:

$$\langle\phi|\psi\rangle = [-i, 1][1, -i]^T = -2i. \tag{4.37}$$

and then divide it by the product of their norm:

$$\frac{-2i}{\sqrt{2} * \sqrt{2}} = -i. \tag{4.38}$$

Equivalently, we can first normalize them, and then take their product:

$$\left\langle \frac{1}{\sqrt{2}}\phi | \frac{1}{\sqrt{2}}\psi \right\rangle = \left[\frac{-i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]\left[\frac{1}{\sqrt{2}}, \frac{-i}{\sqrt{2}}\right]^T = -i. \tag{4.39}$$

The result is, of course, the same. We can concisely indicate it as

$$\frac{\langle\phi|\psi\rangle}{||\phi\rangle| \ ||\psi\rangle|}. \tag{4.40}$$

□

1. Quantum States

■ Keynotes

- We have learned to associate a vector space to a quantum system. The dimension of this space reflects the amount of basic states of the system
- States can be superposed, by adding their representing vectors
- A state is left unchanged if its representing vector is multiplied by a complex scalar
- The state space has a geometry, given by its inner product. This geometry has a physical meaning: it tells us the likelihood for a given state to transition into another one after being measured. States that are orthogonal to one another are mutually exclusive

Review: Lecture 4 (first half)

- Basic Quantum Theory
  1. Quantum States
     - Superposition, linear combination
     - Complex/probability amplitude
     - Ket vector
     - Stern-Gerlach experiment, spin states in x/y/z axis
     - transition amplitude, bra vector

  (Superposition, Complex/probability amplitude, Ket vector: Case 1)

  (Stern-Gerlach experiment, transition amplitude: Case 2)

  2. Observables and measuring
     - Principle 1: measurement is represented as linear operator Ω
     - Principle 2: measurement result are eigenvalues of Ω
     - Principle 3: Unambiguously distinguishable states are orthogonal vectors

2. Observables and measuring

- Specification of a physical system
  - State space: the collection of all states
  - Observable set: the physical quantities that can be observed in each state
- Observable
  - A specific question we pose to the system
- Measuring
  - the process consisting of asking a specific question and receiving a definite answer

2. Observables and measuring

- Classic physics
  - the act of measuring would leave the system in whatever state it already was, at least in principle
  - the result of a measurement on a well-defined state is predictable, i.e., if we know the state with absolute certainty, we can anticipate the value of the observable on that state
- Quantum physics
  - systems do get perturbed and modified as a result of measuring them
  - only the probability of observing specific values can be calculated: measurement is inherently a nondeterministic process

2. Observables and measuring

- Principle 1
  - The observable or measurable quantities of quantum mechanics are represented by linear operators $\mathbf{\Omega}$.
    - In fact, $\mathbf{\Omega}$ must also be Hermitian (see it later)

at these facts. It implies that each observable (σ_x, σ_y, and σ_z) is identified with a specific linear operator in the two-dimensional space of states describing the spin.

2. Observables and measuring

- Principle 2
  - The possible results of a measurement are the eigenvalues of the operator that represents the observable.
  - The collapsed state is the related eigenvector of the operator that represents the observable
  - If the system is in the eigenstate $|\lambda_i\rangle$, the result of a measurement is *guaranteed* to be λ_i .

2. Observables and measuring

- Principle 3
  - Unambiguously distinguishable states are represented by orthogonal vectors.
    - Inner product of two states is a measure of the inability to distinguish them with certainty

Principle 3 is the most interesting. At least I find it so. It speaks of *unambiguously distinct states*, a key idea that we have already encountered. Two states are physically distinct if there is a measurement that can tell them apart without ambiguity. For example, $|u\rangle$ and $|d\rangle$ can be distinguished by

2. Observables and measuring

- Observable operator must be Hermitian
  - Reason 1: since the result of an experiment must be a real number, the eigenvalues of an operator $\boldsymbol{\Omega}$ must also be real
  - Reason 2: the eigenvectors that represent unambiguously distinguishable results must have different eigenvalues, and must also be orthogonal.
  - These conditions are sufficient to prove that $\boldsymbol{\Omega}$ must be Hermitian (try to prove it! Bonus!)

2. Observables and measuring

- Case 1: positions on line
  - Observable is position $\mathbf{P}$
  - $\mathbf{P}$'s operator

$$P = \begin{bmatrix} x_0 & 0 & \cdots & 0 \\ 0 & x_1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{n-1} \end{bmatrix}$$

  - $\mathbf{P}$ acts on basic states

$$\mathbf{P}(|x_i\rangle) = x_i|x_i\rangle$$

Principle 1: observable is Hermitian
Principle 2: eigen-value and –vector
Principle 3: exclusive states are orthogonal

- Position x_i and state $|x_i\rangle$ (Dirac ket notation)
- State associated column vectors
  - E.g. $|x_0\rangle \longmapsto [1, 0, \ldots .0]^T$
    $|x_1\rangle \longmapsto [0, 1, \ldots .0]^T$

2. Observables and measuring

- Case 2: Single-particle spin system
  - z-axis spin operator σ_z

$$\begin{cases} \sigma_z|u\rangle = |u\rangle \\ \sigma_z|d\rangle = -|d\rangle \\ \langle u|d\rangle = 0 \end{cases} \Rightarrow \begin{cases} \begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases} \Rightarrow \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Principles 2 and 3 | Eigenvectors and their eigenvalues | Hermitian operator

2. Observables and measuring

- Case 2: Single-particle spin system
  - x-axis spin operator σ_x

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  - y-axis spin operator σ_y

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$|l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle.$$

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$$

$$|i\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle$$

$$|o\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle$$

2. Observables and measuring

- Some truths about operator
  - Operators are the things we use to calculate eigenvalues and eigenvectors
  - Operators act on state-vectors (which are abstract mathematical objects), not on actual physical systems
  - When an operator acts on a state-vector, it produces a new state-vector

2. Observables and measuring

- A common misconception about the operator
  - When a measurement operator acts on a state-vector, it produces a new state-vector, but that operation is in no way the same as acting on the state with the operator
    - The latter means a state transition $\mathbf{\Psi}|\lambda\rangle$
    - The former means a state collapse $\mathbf{\Omega}|\lambda\rangle$ (only valid when $|\lambda\rangle$ is the eigenvector of $\mathbf{\Omega}$)

感谢弘毅学堂2020级王骏峣同学指出此处“then”->“the”的书写错误

2. Observables and measuring

- A common misconception about the operator
  - Example

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \xrightarrow{\sigma_z} \sigma_z|r\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle \quad (3.21)$$

    - Observation will not leave the system in superposition, the measurement result is either +1 in $|u\rangle$ or -1 in $|d\rangle$
    - The above result allows us to calculate the probabilities of each possible outcome of the measurement

2. Observables and measuring

- Principle 4
  - If $|\psi\rangle$ is the state-vector of a system, and the observable $\mathbf{\Omega}$ is measured, the probability to observe value λ_i is

$$p(\lambda_i) = |\langle\lambda_i|\psi\rangle|^2 = \langle\psi|\lambda_i\rangle\langle\lambda_i|\psi\rangle$$

But, in general, there is no way to tell for certain which of these values will be observed. There is only a probability—let us call it $P(\lambda_i)$—that the outcome will be λ_i. Principle 4 tells us how to calculate that probability, and it is expressed in terms of the overlap of $|A\rangle$ and $|\lambda_i\rangle$. More precisely, the

2. Observables and measuring

- The expected value of observing

$$\langle \mathbf{\Omega} \rangle_{\psi} = \langle \mathbf{\Omega}\psi, \psi \rangle = \langle \psi, \mathbf{\Omega}\psi \rangle$$

This postulate states the following: suppose that

$$\lambda_0, \lambda_1, \ldots, \lambda_{n-1} \tag{4.60}$$

Principle 2

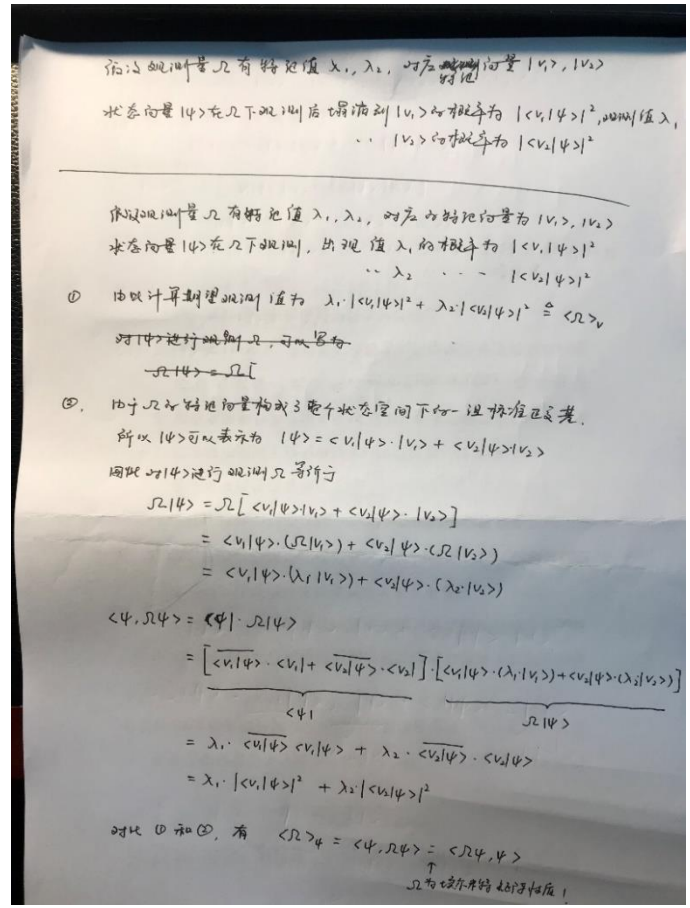
is the list of eigenvalues of Ω. Let us prepare our quantum system so that it is in state $|\psi\rangle$ and let us observe the value of Ω. We are going to obtain one or another of the aforementioned eigenvalues. Now, let us start all over again many times, say, n times, and let us keep track of what was observed each time. At the end of our experiment, the eigenvalue λ_i has been seen p_i times, where $0 \le p_i \le n$ (in statistical jargon, its frequency is p_i/n). Now perform the calculation

$$\lambda_0 \times \frac{p_0}{n} + \lambda_1 \times \frac{p_1}{n} + \cdots + \lambda_{n-1} \times \frac{p_{n-1}}{n}. \tag{4.61}$$

If n is sufficiently large, this number (known in statistics as the estimated expected value of Ω) will be very close to $\langle \Omega\psi, \psi \rangle$.

2. Observables and measuring

- Proof sketch



假设观测量Ω有特征值 λ_1, λ_2，对应特征向量 $|v_1\rangle, |v_2\rangle$

状态向量 $|\psi\rangle$ 在Ω下观测后塌缩到 $|v_1\rangle$ 的概率为 $|\langle v_1|\psi\rangle|^2$，观测值 λ_1，

.. $|v_2\rangle$ 的概率为 $|\langle v_2|\psi\rangle|^2$

假设观测量Ω有特征值 λ_1, λ_2，对应的特征向量为 $|v_1\rangle, |v_2\rangle$

状态向量 $|\psi\rangle$ 在Ω下观测，出现值 λ_1 的概率为 $|\langle v_1|\psi\rangle|^2$

.. λ_2 ... $|\langle v_2|\psi\rangle|^2$

① 由此计算期望观测值为 $\lambda_1 \cdot |\langle v_1|\psi\rangle|^2 + \lambda_2 \cdot |\langle v_2|\psi\rangle|^2 \triangleq \langle \Omega \rangle_\psi$

~~对 $|\psi\rangle$ 进行观测Ω，可以写为~~

~~$\Omega|\psi\rangle = \Omega[$~~

② 由于Ω的特征向量构成了整个状态空间下的一组标准正交基，

所以 $|\psi\rangle$ 可以表示为 $|\psi\rangle = \langle v_1|\psi\rangle \cdot |v_1\rangle + \langle v_2|\psi\rangle \cdot |v_2\rangle$

因此对 $|\psi\rangle$ 进行观测Ω等价于

$$\begin{aligned}\Omega|\psi\rangle &= \Omega[\langle v_1|\psi\rangle \cdot |v_1\rangle + \langle v_2|\psi\rangle \cdot |v_2\rangle] \\ &= \langle v_1|\psi\rangle \cdot (\Omega|v_1\rangle) + \langle v_2|\psi\rangle \cdot (\Omega|v_2\rangle) \\ &= \langle v_1|\psi\rangle \cdot (\lambda_1 |v_1\rangle) + \langle v_2|\psi\rangle \cdot (\lambda_2 \cdot |v_2\rangle)\end{aligned}$$

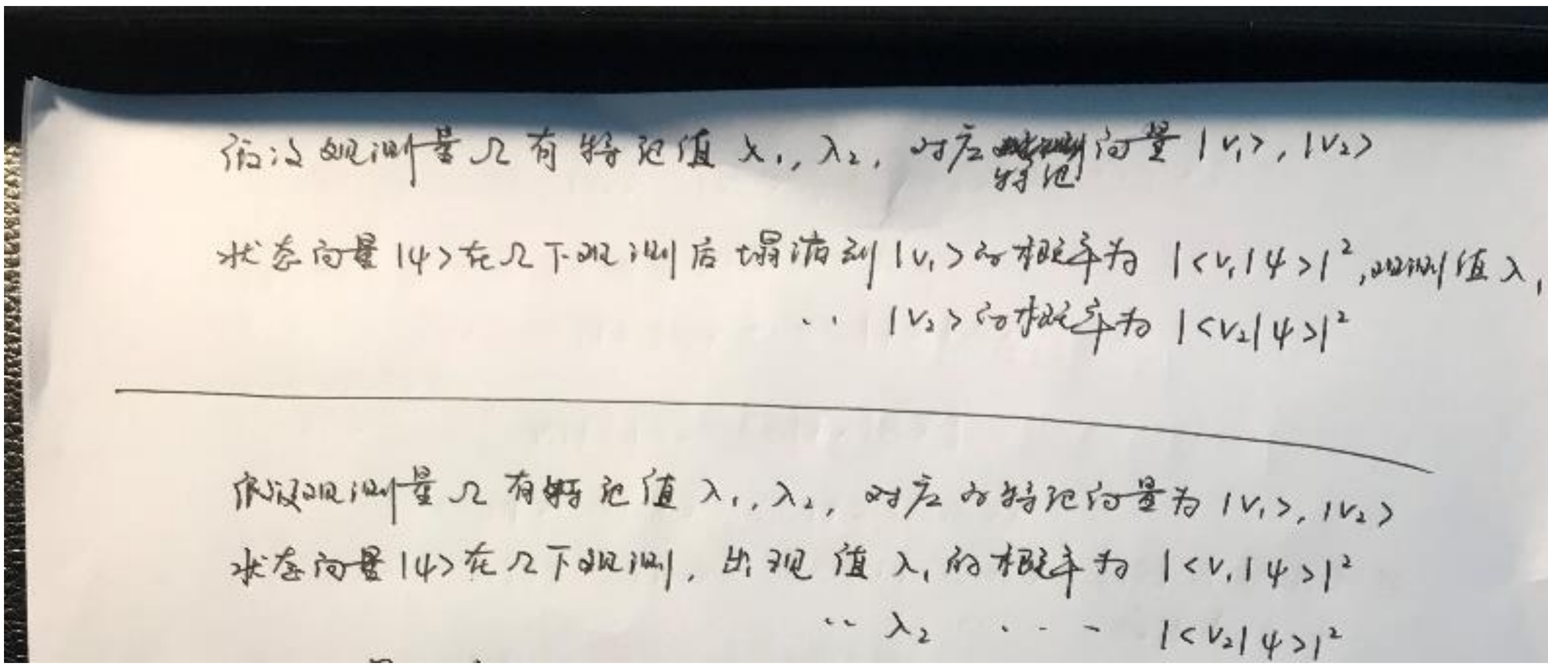
$$\begin{aligned}\langle \psi, \Omega\psi \rangle &= \langle \psi| \cdot \Omega|\psi\rangle \\ &= \underbrace{[\overline{\langle v_1|\psi\rangle} \cdot \langle v_1| + \overline{\langle v_2|\psi\rangle} \cdot \langle v_2|]}_{\langle \psi|} \cdot \underbrace{[\langle v_1|\psi\rangle \cdot (\lambda_1 \cdot |v_1\rangle) + \langle v_2|\psi\rangle \cdot (\lambda_2 |v_2\rangle)]}_{\Omega|\psi\rangle} \\ &= \lambda_1 \cdot \overline{\langle v_1|\psi\rangle} \langle v_1|\psi\rangle + \lambda_2 \cdot \overline{\langle v_2|\psi\rangle} \cdot \langle v_2|\psi\rangle \\ &= \lambda_1 \cdot |\langle v_1|\psi\rangle|^2 + \lambda_2 \cdot |\langle v_2|\psi\rangle|^2\end{aligned}$$

对比①和②，有 $\langle \Omega \rangle_\psi = \langle \psi, \Omega\psi \rangle = \langle \Omega\psi, \psi \rangle$

↑ Ω为埃尔米特矩阵性质！

2. Observables and measuring

- Proof sketch



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〃 〃 $|v_2\rangle$ 的概率为 $|\langle v_2|\psi\rangle|^2$

假设观测量Ω有特征值 λ_1, λ_2，对应的特征向量为 $|v_1\rangle, |v_2\rangle$

状态向量 $|\psi\rangle$ 在Ω下观测，出现值 λ_1 的概率为 $|\langle v_1|\psi\rangle|^2$

〃 λ_2 〃 〃 〃 $|\langle v_2|\psi\rangle|^2$

2. Observables and measuring

■ Proof sketch

②. 由于Ω的特征向量构成了整个状态空间下的一组标准正交基.

所以 $|\psi\rangle$ 可以表示为 $|\psi\rangle = \langle v_1|\psi\rangle\cdot|v_1\rangle + \langle v_2|\psi\rangle\cdot|v_2\rangle$

因此对 $|\psi\rangle$ 进行观测Ω等价于

$$\Omega|\psi\rangle = \Omega\left[\langle v_1|\psi\rangle\cdot|v_1\rangle + \langle v_2|\psi\rangle\cdot|v_2\rangle\right]$$

$$= \langle v_1|\psi\rangle\cdot(\Omega|v_1\rangle) + \langle v_2|\psi\rangle\cdot(\Omega|v_2\rangle)$$

$$= \langle v_1|\psi\rangle\cdot(\lambda_1|v_1\rangle) + \langle v_2|\psi\rangle\cdot(\lambda_2\cdot|v_2\rangle)$$

2. Observables and measuring

- Proof sketch

$$\langle \psi, \Omega\psi \rangle = \langle \psi | \cdot \Omega | \psi \rangle$$

$$= \underbrace{\left[\overline{\langle v_1 | \psi \rangle} \cdot \langle v_1 | + \overline{\langle v_2 | \psi \rangle} \cdot \langle v_2 | \right]}_{\langle \psi |} \cdot \underbrace{\left[\langle v_1 | \psi \rangle \cdot (\lambda_1 \cdot | v_1 \rangle) + \langle v_2 | \psi \rangle \cdot (\lambda_2 | v_2 \rangle) \right]}_{\Omega | \psi \rangle}$$

$$= \lambda_1 \cdot \overline{\langle v_1 | \psi \rangle} \langle v_1 | \psi \rangle + \lambda_2 \cdot \overline{\langle v_2 | \psi \rangle} \cdot \langle v_2 | \psi \rangle$$

$$= \lambda_1 \cdot | \langle v_1 | \psi \rangle |^2 + \lambda_2 \cdot | \langle v_2 | \psi \rangle |^2$$

对比①和②，有 $\langle \Omega \rangle_\psi = \langle \psi, \Omega\psi \rangle = \langle \Omega\psi, \psi \rangle$

↑ Ω为埃尔米特矩阵性质！

2. Observables and measuring

- The expected value of observing

Example 4.2.4 Let us calculate the expected value of the position operator on an arbitrary normalized state vector: let

$$|\psi\rangle = c_0|x_0\rangle + c_1|x_1\rangle + \cdots + c_{n-1}|x_{n-1}\rangle \tag{4.62}$$

be our state vector and

$$\langle P\psi, \psi\rangle = |c_0|^2 \times x_0 + |c_1|^2 \times x_1 + \cdots + |c_{n-1}|^2 \times x_{n-1}, \tag{4.63}$$

remember: $\mathbf{P}(|x_i\rangle) = x_i|x_i\rangle$

where

$$|c_0|^2 + |c_1|^2 + \cdots + |c_{n-1}|^2 = 1. \tag{4.64}$$

In particular, if $|\psi\rangle$ happens to be just $|x_i\rangle$, we simply get x_i (verify it!). In other words, the expected value of position on any of its eigenvectors $|x_i\rangle$ is the corresponding position x_i on the line. □

2. Observables and measuring

- The expected value of observing

Example 4.2.5 Let $|\psi\rangle = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}i\right]^T$ and $\Omega = \begin{bmatrix} 1 & -i \\ i & 2 \end{bmatrix}$.

Let us calculate $\Omega(|\psi\rangle)$:

$$\Omega(|\psi\rangle) = \begin{bmatrix} 1 & -i \\ i & 2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2}i \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \frac{3}{2}\sqrt{2}i \end{bmatrix}. \tag{4.65}$$

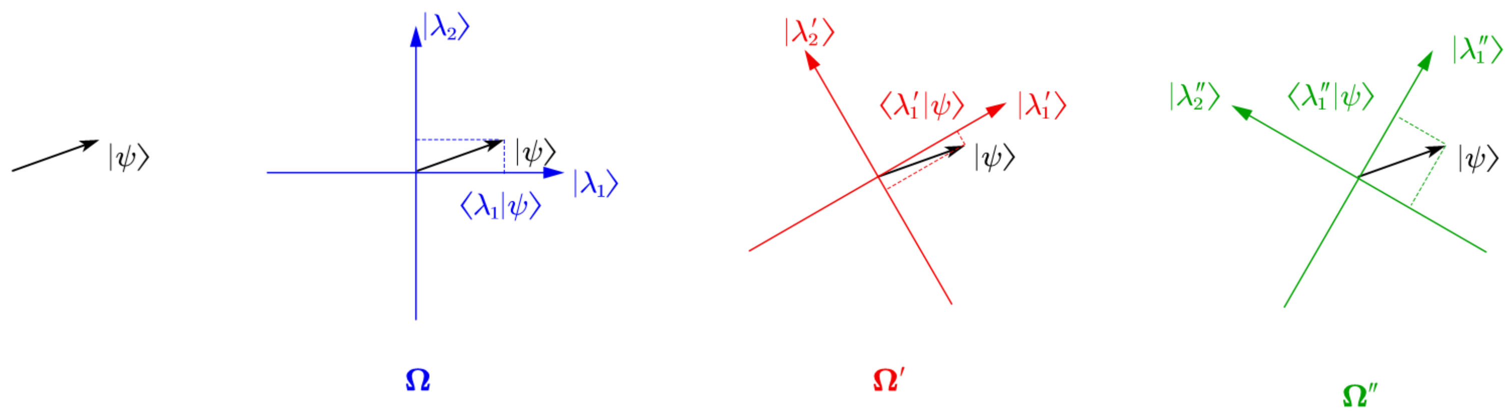
The bra associated with $\Omega|\psi\rangle$ is $\left[\sqrt{2}, -\frac{3}{2}\sqrt{2}i\right]$. The scalar product $\langle\Omega\psi|\psi\rangle$, i.e., the average value of Ω on $|\psi\rangle$, is thus equal to

$$\left[\sqrt{2}, -\frac{3}{2}\sqrt{2}i\right]\left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}i\right]^T = 2.5. \tag{4.66}$$

□

2. Observables and measuring

- What happens after measuring?
  - Get an answer λ_i with probability $p_i = \langle\psi|\lambda_i\rangle\langle\lambda_i|\psi\rangle$
  - The system transitions to the corresponding eigenvector $|\lambda_i\rangle$

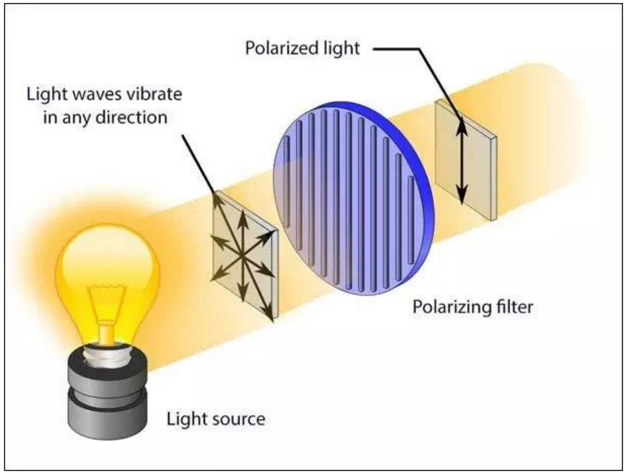


2. Observables and measuring

- What happens after measuring?
  - Get an answer λ
  - The system transitions to the corresponding eigenvector $|\lambda\rangle$
- What is going to happen if we conduct the same measurement immediately thereafter?
  - Exactly the same answer, and stay where it is

2. Observables and measuring

- How about multiple observing?
  - Order matters
- Experiment: light passing through polarization sheet



2. Observables and measuring

- How about multiple observing?
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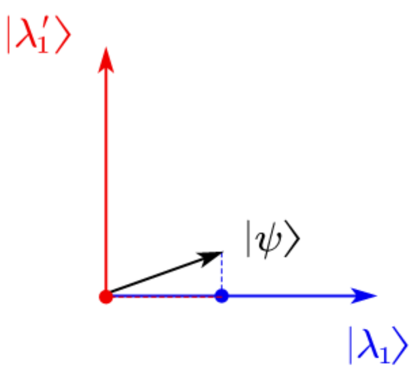
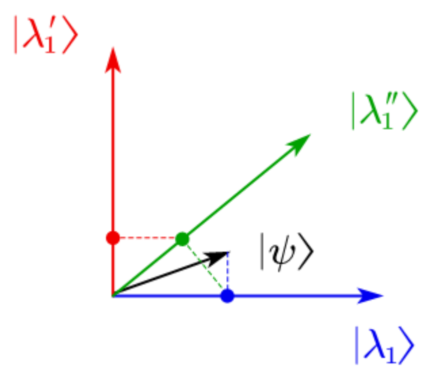
- Experiment: light passing through polarization sheet





2. Observables and measuring

- Keynotes
  - Observables are represented by Hermitian operators. The result of an observation is always an eigenvalue of the Hermitian.
  - The end state of the measurement of an observable is always one of its eigenvectors
  - The probability for an initial state to collapse into an eigenvector of the observable is given by the length squared of the projection
  - The expression $\langle \mathbf{\Omega} \rangle_{\psi}$ represents the expected value of observing $\mathbf{\Omega}$ on $|\psi\rangle$
  - When we measure several observables, the order of measurements matters (because observation will change the state)

3. Dynamics

- Principle 5
  - The evolution of a quantum system (that is not a measurement) is given by a unitary operator or transformation
    - $|\boldsymbol{\psi}(t+1)\rangle = \mathbf{U}|\boldsymbol{\psi}(t)\rangle$

State of the system at time *t+1*

a unitary matrix that represents a unitary operator

State of the system at time *t*

3. Dynamics

- Features

An important feature of unitary transformations is that they are closed under composition and inverse, i.e., the product of two arbitrary unitary matrices is unitary, and the inverse of a unitary transformation is also unitary. Finally, there is a multiplicative identity, namely, the identity operator itself (which is trivially unitary). In math jargon, one says that the set of unitary transformations constitutes a **group of transformations** with respect to composition.

3. Dynamics

■ The process of quantum computing

- Prepare an initial state $|\psi\rangle$
- Apply a sequence of unitary operators to the state

$$|\psi\rangle \underset{\mathfrak{U}[t_0]^\dagger}{\overset{\mathfrak{U}[t_0]}{\rightleftarrows}} \mathfrak{U}[t_0]|\psi\rangle \underset{\mathfrak{U}[t_1]^\dagger}{\overset{\mathfrak{U}[t_1]}{\rightleftarrows}} \mathfrak{U}[t_1]\mathfrak{U}[t_0]|\psi\rangle \underset{\mathfrak{U}[t_2]^\dagger}{\overset{\mathfrak{U}[t_2]}{\rightleftarrows}} \mathfrak{U}[t_2]\mathfrak{U}[t_1]\mathfrak{U}[t_0]|\psi\rangle$$

$$\rightleftarrows \cdots \rightleftarrows \mathfrak{U}[t_{n-1}]\mathfrak{U}[t_{n-2}]\cdots\mathfrak{U}[t_0]|\psi\rangle. \qquad (4.94)$$

- Measure the output and get a final state

3. Dynamics

- Schrödinger equation

$$\frac{|\boldsymbol{\psi}(t+\delta t)\rangle - |\boldsymbol{\psi}(t)\rangle}{\delta t} = -i\frac{2\pi}{\hbar}\mathcal{H}|\boldsymbol{\psi}(t)\rangle$$

  - $\mathcal{H}$ is the Hamiltonian of the system

of an isolated system is preserved throughout its evolution.[14] Energy is an observable, and therefore for a concrete quantum system it is possible to write down a hermitian matrix representing it (this expression will of course vary from system to system). This observable is called the **hamiltonian** of the system, indicated by $\mathcal{H}$ in Equation (4.96).

The Schrödinger equation states that the rate of variation of the state vector $|\psi(t)\rangle$ with respect to time at the instant t is equal (up to the scalar factor $\frac{2\pi}{\hbar}$) to $|\psi(t)\rangle$ multiplied by the operator $-i * \mathcal{H}$. By solving the equation with some initial conditions one is able to determine the evolution of the system over time.

3. Dynamics

- Keynotes
  - Quantum dynamics is given by unitary transformations
  - Unitary transformations are invertible; thus, all closed system dynamics are reversible in time (as long as no measurement is involved)
  - The concrete dynamics is given by the Schrödinger equation, which determines the evolution of a quantum system whenever its Hamiltonian is specified

Conclusion

1. Quantum States
   - Complex/probability/transition amplitudes
2. Observables & Measuring
   - The observables are represented by Hermitian matrices
   - The possible results of a measurement are the eigenvalues of the observable matrices. If the system is in the eigenstate, the measurement result is guaranteed to be the related eigenvalue
   - Unambiguously distinguishable states are represented by orthogonal vectors
   - Observing probability is the modulus square of the probability amplitude
3. Dynamics
   - The evolution of a quantum system is given by a unitary matrix